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Europ. J. Combinatorics (1990) **11**, 7–13

Not All Graphs are Segment T-graphs

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Given two line segments in the plane, we say they are in *T-position* if the line containing one of the segments intersects the other segment. A *segment T-graph* has its vertices in one-to-one correspondence with pairwise disjoint planar line segments so that two vertices are adjacent iff they are in T-position. We give two proofs that not all graphs are segment T-graphs.

1. INTRODUCTION

Certain results on common transversals for families of disjoint segments in the plane (see [6]), lead to the following definitions. Given two planar line segments, $S_1 = P_1Q_1$ and $S_2 = P_2Q_2$, we say that S_1 *cuts* (or *shoots*) S_2 (notation: $S_1 \to S_2$) provided that the line through S_1 intersects S_2. We say that S_1 and S_2 are in *T-position* provided that $S_1 \to S_2$ or $S_2 \to S_1$. Note that this is equivalent to the statement that the end points of one segment are on opposite sides of the line through the other segment.

A (finite, simple) graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ is called a *segment T-graph* provided that one can assign to each of its vertices v_i a segment S_i in the plane so that all segments are pairwise disjoint and v_i is adjacent to v_j (notation: $v_i \sim v_j$) iff S_i and S_j are in T-position.

Katchalski asked whether all graphs are segment T-graphs. Our main result is that this is not the case:

THEOREM 1.1. *Not all graphs are segment T-graphs.*

We give two proofs of this result, one involving the geometry of convex sets and Ramsey theory, and the other based on a theorem of Warren [9] from real algebraic geometry.

2. ENUMERATION OF STABBING LINES

Our first proof is based on the following enumerative result concerning lines and convex sets in the plane.

Let $\mathcal{A}$ denote a set of n disjoint compact convex sets in the plane. Let $\mathcal{B} \subset \mathcal{A}$. We say that a line L stabs $\mathcal{B}$ provided that L intersects every set in $\mathcal{B}$ but no set in $\mathcal{A} - \mathcal{B}$, that is $\mathcal{B} = \{A \in \mathcal{A} : A \cap L \neq \emptyset\}$. Denote by $\Sigma(\mathcal{A})$ the set of all subsets $\mathcal{B} \subset \mathcal{A}$ for which there exists a stabbing line. Finally, put $\sigma(n) = \max\{|\Sigma(\mathcal{A})| : |\mathcal{A}| = n\}$.

THEOREM 2.1. *There exists a constant $c > 0$ such that*

$$4\binom{n}{2} - cn \leq \sigma(n) \leq 4\binom{n}{2} + n.$$

PROOF. We begin with the lower bound. Let $\mathcal{A}$ consist of $n-2$ horizontal line segments $S_1, \ldots, S_{n-2}$ and two vertical segments S_L, S_R, as shown in Figure 1.

Note that the segments $S_1, \ldots, S_{n-2}$ are horizontal. Their left (respectively, right) end points lie along a strictly convex curve. The segments S_L and S_R are vertical. Their

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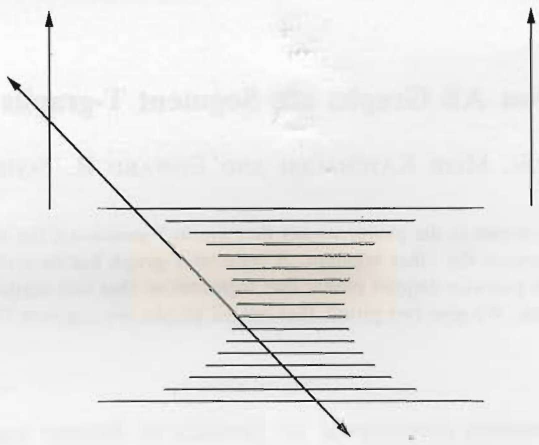

FIGURE 1. A family with $|\Sigma(\mathscr{A})| \geq 4\binom{n}{2} - cn$.

lower end points are at the same height as the topmost horizontal segment (S_1) and they are very tall (as indicated by the arrows in the figure). In addition, S_L (respectively, S_R) is very far to the left (respectively, right) of the horizontal segments: in the figure we have drawn them close to the horizontal segments for the sake of clarity.

For each pair k, l (with $1 \leq k < l \leq n-2$) there exists subsets $\mathscr{B} \subset \mathscr{A}$ of each of the following types which is stabbed by some line:

(1) $\{S_i: 1 \leq i \leq k \text{ or } l \leq i \leq n-2\} \cup \{S_R\}$;
(2) $\{S_i: 1 \leq i \leq k \text{ or } l \leq i \leq n-2\} \cup \{S_L\}$;
(3) $\{S_i: k \leq i \leq l\} \cup \{S_R\}$;
(4) $\{S_i: k \leq i \leq l\} \cup \{S_L\}$.

This gives

$$4\binom{n-2}{2} = 4\binom{n}{2} - O(n)$$

subsets which are stabbed by some line. (Note that, in fact, there are some more subsets stabbed by lines, but the ones mentioned above suffice for our purposes.)

Now we consider the upper bound. For each pair of sets in $\mathscr{A}$ there exist four mutual support lines; two (called support lines of type one) in which the sets are on the same side of the line and two (called support lines of type two) in which the sets are on opposite sides (see Figure 2).

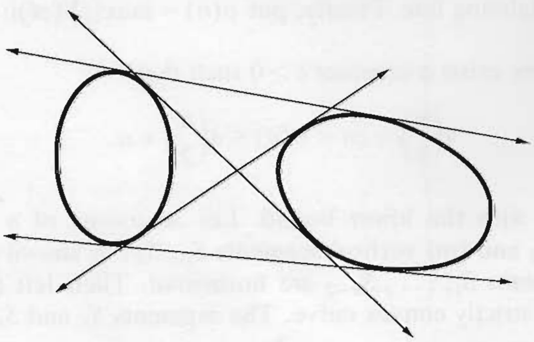

FIGURE 2. Four mutual support lines.

An *ε-expansion* of a planar set A is a superset of A containing all points in the plane which are no further than a distance ε from A. The expansion of a compact convex set is still compact and convex. Observe that since our sets are compact, they may be expanded by a sufficiently small amount with the expansion perhaps adding sets to $\Sigma(\mathscr{A})$, but never removing sets from $\Sigma(\mathscr{A})$. By expanding the sets in $\mathscr{A}$ by varying small amounts, we lose no generality in assuming that all support lines are distinct and no two are parallel.

For each $\mathscr{B} \in \Sigma(\mathscr{A})$ of cardinality at least two, we find two 'extreme' lines l_0 and l_1 in the following manner. Choose, arbitrarily, a line separating two members of $\mathscr{B}$ and choose a co-ordinate system in which this line is the x-axis. Clearly, in this co-ordinate system, any line that stabs $\mathscr{B}$ contains points above and below the x-axis. Orient each such line l from bottom to top, and let $a(l)$ be the angle between the positive x-axis and the oriented line l. Define:

$$a_0 = \inf\{a(l): l \text{ stabs } \mathscr{B}\} \qquad \text{and} \qquad a_1 = \sup\{a(l): l \text{ stabs } \mathscr{B}\}.$$

Let D be a large disc containing all the sets in $\mathscr{A}$. By the definition of a_0, a_1 and by compactness there are lines $\{p_i\}$ and $\{q_i\}$ that stab $\mathscr{B}$ so that $p_i \cap D$ converges to $l_0 \cap D$ and $q_i \cap D$ converges to $l_1 \cap D$, for some two lines l_0 and l_1 that satisfy $a(l_0) = a_0$ and $a(l_1) = a_1$. Clearly, as all the sets are compact, both l_0 and l_1 intersect all the members of $\mathscr{B}$ (and possibly some other sets as well). One can easily check that each of l_0 and l_1 is one of the $4\binom{n}{2}$ support lines defined by pairs of sets of $\mathscr{A}$. Moreover, a support line of type one defined by the sets A and B can be obtained only if either ($A \in \mathscr{B}$ and $B \notin \mathscr{B}$) or ($A \notin \mathscr{B}$ and $B \in \mathscr{B}$). This is because otherwise the line can be slightly rotated to any direction to obtain another line that stabs $\mathscr{B}$, contradicting the definition of a_0 and a_1. Similarly, a support line of type two defined by the sets A and B can be obtained only if both belong to $\mathscr{B}$ or if both do not belong to $\mathscr{B}$. Consequently, every support line is obtained at most twice, and as for each $\mathscr{B} \in \Sigma(\mathscr{A})$ (with $|\mathscr{B}| \geq 2$) we obtain two extreme lines: we conclude that the number of sets of cardinality at least two in $\Sigma(\mathscr{A})$ is at most

$$\frac{2 \cdot 4\binom{n}{2}}{2} = 4\binom{n}{2}.$$

Finally, one adds n to the upper bound to account for the $\mathscr{B}$ of cardinality 1. This completes the proof. □

REMARK 2.2. The main result of Brisztriczky and Pach in [2] is that if (i) $\mathscr{A}$ is a family of m pairwise disjoint compact convex sets in the plane, (ii) $n \geq 3$ and (iii) for every subset $\mathscr{B} \subset \mathscr{A}$ of cardinality $|\mathscr{B}| = n \geq 3$ we have $\mathscr{B} \in \Sigma(\mathscr{A})$, then $m \leq n + 46$. By Theorem 2.1 (and its proof) this can be improved to $m \leq n + 2$ for any $n \geq 12$. Indeed, if $m \geq n + 3$ then, by the proof of Theorem 2.1,

$$\binom{n+3}{3} \leq 4\binom{n+3}{2}$$

(since the number of sets of cardinality n in $\Sigma(\mathscr{A})$ for $\mathscr{A}$ of cardinality $n + 3$ cannot exceed $4\binom{n+3}{2}$). This gives $n \leq 11$. The bound for smaller values of n can also be improved, similarly, using the fact that $\binom{l}{n} \leq 4\binom{l}{3}$ for every $l \leq m$.

3. RAMSEY THEORETIC PROOF OF THEOREM 1.1

We use the following result from Ramsey theory, due to Nesetril and Rödl ([8]; see also [5], p. 103).

THEOREM 3.1 *For every positive integer r and every bipartite graph G, there exists a bipartite graph H such that for every r-coloring of the edges of H, there exists in H an* induced *copy of G in which all the edges have the same color (induced monochromatic copy of G).*

FIRST PROOF OF THEOREM 1.1. Let G_0 be the bipartite graph $(X \cup Y, E)$, where

$$X = \{x_1, x_2, \dots, x_n\}$$

and

$$Y = \{y_A : A \subset \{1, \dots, n\}, A \neq \emptyset\}.$$

Thus $|X| = n$ and $|Y| = 2^n - 1$; the elements of Y are indexed by the non-empty subsets of $\{1, \dots, n\}$. For $x_i \in X$ and $y_A \in Y$ put

$$x_i \sim y_A \Leftrightarrow i \in A.$$

Let G be formed by taking two disjoint copies of G_0 and adding a single edge from one specific vertex in one copy of G_0 to its 'twin' in the other copy. Note that G is bipartite.

Finally, let H be a bipartite graph (the existence of which is guaranteed by Theorem 3.1) such that any two-coloring of the edges of H contains an induced monochromatic copy of G. We claim that H is not a segment T-graph.

Suppose H were a segment T-graph and fix a representation of H by segments in the plane. Since H is bipartite, let $U \cup W$ be a partition of the vertices of H into independent sets. Color the edges of H as follows. For $uw \in E(H)$ with $u \in U$ and $w \in W$, color uw RED if $S_u \to S_w$; otherwise, (if $S_w \to S_u$) color uw BLUE.

By construction, H contains an induced copy of G in which all edges are (without loss of generality) RED. Thus each U-vertex of G shoots its W-neighbors. Thus in one of the two G_0's in G, it must be the case that each y-vertex shoots its x-neighbors.

Let $\mathcal{A} = \{S_1, \dots, S_n\}$ denote the set of line segments assigned to the x_i's in this particular copy of G_0. Note that for each non-empty subset $\mathcal{B} \subset \mathcal{A}$ there is a line which stabs $\mathcal{B}$, namely the line which corresponds to the appropriate y-vertex in G_0. Thus, $|\Sigma(\mathcal{A})| = 2^n - 1$, but by Theorem 2.1, $|\Sigma(\mathcal{A})| \leq 4\binom{n}{2} + n$, which is impossible for $n \geq 7$. □

Note that the graph H, the existence of which is described above, can be explicitly constructed by using the proof in [8]. However, it will be a huge graph, much larger than the one produced in our second proof.

4. ALGEBRAIC PROOF OF THEOREM 1.1

This second proof is based on results concerning the sign patterns for polynomials. Let $p_1, \dots, p_m$ be m real polynomials in l variables. If $x \in \mathbf{R}^l$ then $[\mathrm{sgn}(p_1(x)), \dots, \mathrm{sgn}(p_m(x))]$ is the sign pattern of $p_1, \dots, p_m$ at x. Denote the total number of sign patterns that consist of ± 1 terms by $s(p_1, \dots, p_m)$. The following result is due to Warren [9]. (See also [1] for a similar result with a somewhat different proof.)

THEOREM 4.1. *If $p_1, \dots, p_m$ are as above, $m \geq l$ and each p_i is of degree at most $d \geq 1$, then*

$$s(p_1, \dots, p_m) \leq (4edm/l)^l.$$

Let $S_1 = P_1Q_1$ and $S_2 = P_2Q_2$ be two line segments in the plane with $P_i = (a_i b_i)$ and

$Q_i = (c_i, d_i)$ $(i = 1, 2)$ the co-ordinates of their end points. Let $P_{S_1S_2}$ be the following polynomial of degree 4 in the variables a_i, b_i, c_i, d_i:

$$P_{S_1S_2} = \det\left\{\begin{bmatrix} 1 & a_1 & b_1 \\ 1 & c_1 & d_1 \\ 1 & a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} 1 & a_1 & b_1 \\ 1 & c_1 & d_1 \\ 1 & c_2 & d_2 \end{bmatrix}\right\}.$$

One can check easily that $P_{S_1S_2} < 0$ iff $S_1 \to S_2$. Therefore, the segment T-graph corresponding to the n segments $S_1, \ldots, S_n$ is determined by the sign pattern of the sequence of the $2\binom{n}{2}$ polynomials $\{P_{S_iS_j}: i \neq j\}$. Note that the number of variables here is $l = 4n$: hence we deduce, from Theorem 4.1,

THEOREM 4.2. *The total number of segment T-graphs on n labeled vertices is at most* $[4e(n-1)]^{4n}$.

REMARK 4.3. The estimate given in Theorem 4.2 holds even if we do not insist that the line segments in the definition of a segment T-graph be pairwise disjoint.

Furthermore, the estimate given in Theorem 4.2 is sharp, in the sense that the logarithm of the number of segment T-graphs on n labeled vertices is $(1 + o(1))4n \log n$. Indeed, let $m = n/\log n$ and take m vertical and m horizontal segments, situating above and to the right respectively of the unit square in the plane. Notice that these define a partition of the unit square into $(m-1) \times (m-1)$ cells, as shown in Figure 3. (In addition, there are m more segments which we will place very far to the 'southwest' of the unit square. More on this in a moment.) For each of the remaining $n - 3m$ vertices, we place disjoint, non-collinear segments, each with slope $+1$, with end points in the unit square. Observe that for each segment, there are at least cm^3 choices for the cells of its end points (for some constant c). Further, for each selection of end point cells, we have determined which of the $2m$ vertical and horizontal segments shoot the segment in question. After placing the $n - 3m$ segments, which are compact, we note that the slope of each can be changed by up to some small amount ε, without disturbing the representation. Finally, we place the last m segments very far to the 'southwest' of the unit square. These segments are disjoint, collinear and have slope -1. Note that by slight rotation of any of the $n - 3m$ segments with slope $+1$, we can have it shoot whichever of the m southwestern segments we choose. Thus, the number of segment T-graphs on n labeled vertices is bounded below by $(cm^4)^{n-3m}$, the

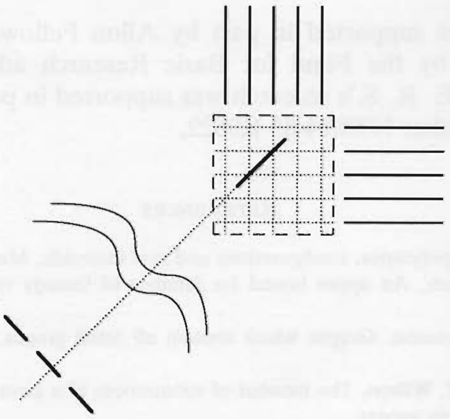

FIGURE 3.

logarithm of which equals

$$(n-3m)\log(cm^4) = (n-3n/\log n)(\log c + 4\log n - 4\log\log n) = [4+o(1)]n\log n.$$

SECOND PROOF OF THEOREM 1.1. From Theorem 4.2 we observe that if

$$2^{\binom{n}{2}} > [4e(n-1)]^{4n}$$

then there is a graph on n vertices which is not a segment T-graph. This inequality is clearly satisfied for n large enough ($n \geq 100$ will do). □

COROLLARY 4.4. *If*

$$2^{\binom{n}{2}} > [4e(n-1)]^{4n}$$

and $G=(V, E)$ is a graph which contains every graph on n vertices as an induced subgraph, then G is not a segment T-graph.

Moon [7] gives an explicit construction of a graph G with $O(n2^{n/2})$ vertices that contains every graph on n vertices as an induced subgraph. This gives an explicit graph with fewer than 2^{60} vertices which is not a segment T-graph.

Somewhat larger examples can also be constructed using the main result of [4]. A distinguished example is the family of Paley graphs. Suppose $p \equiv 1 \pmod 4$ is a prime and let G_p denote the graph the vertices of which are all residue classes mod p in which i and j are adjacent iff $i-j$ is a quadratic residue modulo p. By the main result of [3], if $p > n^2 2^{2n-2}$ then G_p contains every graph on n vertices as an induced subgraph.

COROLLARY 4.5. *If $p > 2^{300}$ is a prime with $p \equiv 1 \pmod 4$, then G_p is not a segment T-graph.*

REMARK 4.6. As our second proof is based on a counting argument it implies that any class of graphs that contain sufficiently many labeled members on n vertices contains a member which is not a segment T-graph. Thus, for example, for all sufficiently large n, there is a bipartite 10 000-regular non-segment T-graph on n vertices with girth at least 10 000. This does not follow from the first proof.

ACKNOWLEDGMENTS

N. A.'s research was supported in part by Allon Fellowship, by a Bat Sheva de Rothschild grant and by the Fund for Basic Research administered by the Israel Academy of Sciences. E. R. S.'s research was supported in part by the Office of Naval Research, contract number N00014-85-K0622.

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Received 8 December 1987 and accepted in revised form 5 July 1989

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